Radiative Corrections to the Casimir Energy

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Abstract: The lowest radiative correction to the Casimir energy density between two parallel plates is calculated using effective field theory. Since the correlators of the electromagnetic field diverge near the plates, the regularized energy density is also divergent. However, the regularized integral of the energy density is finite and varies with the plate separation L as $1/L^7$. This apparently paradoxical situation is analyzed in an equivalent, but more transparent theory of a massless scalar field in 1+1 dimensions confined to a line element of length L and satisfying Dirichlet boundary conditions.

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The remarkable prediction of quantum electrodynamics (QED) by Casimir[1] that two parallel metallic plates in vacuum and with separation L should be attracted by a force per unit area equal to $\pi^2/240L^4$, has very recently been verified experimentally by Lamoreaux[2] with much higher precision than before[3]. In fact, the experiment is so accurate that with further improvements it can hopefully also demonstrate the corrections due to finite-temperature effects[4][5][6][7]. It would then be a new macroscopic system where the intricacies of quantum field theory can be studied.

Despite an enormous literature on the subject (see for example the review article[8] and a recent textbook[9]) very little has been done to investigate the radiative corrections to the this effect based on QED which is basically an interacting theory. It is still not clear at which order in the fine structure constant α the first correction will appear and how it will vary with the plate separation. One of the first calculations were done by Bordag, Robaschik and Wieczorek[10] who found a correction proportional to α/L^5 in lowest order QED perturbation theory using a photon propagator which is appropriately modified due to the plate confinement. Using instead periodic boundary conditions, Xue found that the corresponding radiative corrections were exponentially small[11].

Instead of using QED, we will here use the modern approach of effective field theory [12]. (A pedagogic introduction has been given by Kaplan[13]). The energy scale relevant for the Casimir effect is much smaller than the electron mass m. We can thus integrate out the heavy electron modes from the full Lagrangian and thus obtain a low-energy effective Lagrangian for the interacting photon field. The interactions are due to coupling of the photon to virtual electron-positron pairs in the vacuum and they become local in the low-energy limit. To lowest order in α the first effective coupling is just the Uehling correction[14] which modifies the photon propagator at short distances. It gives rise to

the effective photon Lagrangian

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{\alpha}{60\pi m^2}F_{\mu\nu}\Box F^{\mu\nu} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. If this new interaction contributes, we see that it will modify the Casimir energy density by a term proportional with $\alpha/m^2 L^6$. Barring some extra factors of m from the evaluation of the corresponding Feynman diagram or from the integration over the volume between the plates, we see that we are not able to reproduce the result of Bordag *et al* [10]. However, the corresponding Feynman diagram which is a photon loop with the Uehling correction in the propagator, is in fact zero after regularization. This is most easily seen by using the equation of motion $\Box F_{\mu\nu} = 0$ for the photon following from the leading, Maxwell part of the Lagrangian (1). One might perhaps argue that the classical equation of motion only applies to free, i.e. on-shell photons and not to virtual particles inside a Feynman diagram. But a more careful analysis[15] shows that the Uehling interaction does not contribute to the vacuum energy.

The lowest order correction must thus arise from higher order interactions, i.e. of dimensions eight or more. Among these we first have the next order Uehling term proportional with $F_{\mu\nu}\Box^2 F^{\mu\nu}$. But this will again not contribute for the same reasons as above. But to the same order we also have the Euler-Heisenberg interaction[16]

$$\mathcal{L}_{EH} = \frac{2\alpha^2}{45m^4}\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right] \tag{2}$$

which describes the interaction of four photons via coupling to a virtual electron loop in the vacuum. It has previously been used by Scharnhorst and Barton to investigate the propagation of light in the same geometry of two parallel plates[17]. For dimensional reasons it will thus give an additional Casimir force which varies with the plate separation like $1/L^8$.

In lowest order perturbation theory the correction to the vacuum energy density is given by the expectation value $\langle\mathcal{L}_{EH}\rangle$. For its evaluation we need the correlators $\langle\mathbf{E}^2\rangle$ and $\langle\mathbf{B}^2\rangle$ which give the fluctuations of the free electric and magnetic fields between the plates. The modes must satisfy the metallic boundary conditions $\mathbf{n}\wedge\mathbf{E} = \mathbf{n}\cdot\mathbf{B} = 0$ at the plates and have previously been obtained by Lütken and Ravndal by quantizing the field in the Coulomb gauge and expanding it in electromagnetic multipoles[18]. They can also be obtained from the coincidence limit of the photon propagator constructed by Bordag *et al* [10] in the Lorentz gauge[15]. The results can be summarized as

$$\langle\mathbf{E}^2\rangle = -\frac{\pi^2}{16L^4}\left(\frac{1}{45} - F(\theta)\right), \qquad \langle\mathbf{B}^2\rangle = -\frac{\pi^2}{16L^4}\left(\frac{1}{45} + F(\theta)\right) \tag{3}$$

where $\theta = z\pi/L$ is the scaled distance between the plates and the function

$$F(\theta) = -\frac{1}{2}\frac{d^3}{d\theta^3}\cot\theta = \frac{3}{\sin^4\theta} - \frac{2}{\sin^2\theta} \tag{4}$$

gives the position dependence of the fluctuations. However, when one calculates the energy density $\mathcal{E} = \frac{1}{2}(\langle \mathbf{E}^2 \rangle + \langle \mathbf{B}^2 \rangle)$, it cancels out and gives $\mathcal{E} = -\pi^2/720L^4$ which is the standard result for the Casimir energy per unit volume[1]. When one gets very near one of the plates, the fluctuations become large. Near the left plate at $z = 0$ they increase like

$$\langle \mathbf{E}^2 \rangle = \frac{3}{16\pi^2 z^4}, \qquad \langle \mathbf{B}^2 \rangle = -\frac{3}{16\pi^2 z^4} \tag{5}$$

This represents also the fluctuations near any curved surface since it will always look plane when one gets near enough[19].

With the free field electromagnetic correlators established between the plates, it is now straightforward to calculate the lowest order correction from the Euler-Heisenberg interaction (2). In terms of Feynman diagrams, it is simply given by the contribution from the product of two photon loops,

$$\Delta\mathcal{E} = -\frac{\alpha^2 \pi^4}{2^7 3^3 5 m^4 L^8} \left[\frac{11}{225} + 9F^2(\theta) \right] \tag{6}$$

This additional term in the Casimir energy density is now seen to diverge when one gets near one of the plates. The divergence is directly related to the the corresponding divergence in the correlators. We are now faced with the problem that when calculating the resulting correction to the Casimir force, we need the total vacuum energy between the plates which is obviously also divergent.

This is physically meaningless, but not really a new situation. For instance, when one calculates the regularized vacuum energy density for free photons inside a spherical cavity[20], it diverges near the surface as in (5) and the integrated energy is thus also divergent. However, if one instead calculates the total energy which afterwards is regularized, one obtains a finite result[21][22][23]. That the processes of regularization and integration do not commute in general for these kinds of problems have been discussed by Deutsch and Candelas[19][24].

In our case with interacting fields we can obtain the total unregularized energy by integrating the unregularized energy density. The resulting expression can then be regularized and we have a finite, total Casimir energy. Since the calculation is somewhat cumbersome, we can illustrate it by neglecting the uninteresting degrees of freedom parallel to the plates and consider instead the simpler theory of a massless field in 1+1 dimensions with Lagrangian $\mathcal{L} = (1/2)[(\partial_t \phi)^2 - (\partial_z \phi)^2]$. Here $\partial_t \phi$ can be taken to be the electric field and $\partial_z \phi$ to be the magnetic field. We can thus continue to talk about the electric $\mathcal{E}_E = (1/2)\langle (\partial_t \phi)^2 \rangle$ and magnetic $\mathcal{E}_B = (1/2)\langle (\partial_t \phi)^2 \rangle$ contributions to the vacuum energy density. Imposing the Dirichlet condition that the field vanishes at the boundaries, the normalized eigenmodes of the field are

$$\phi_n(z) = \sqrt{\frac{2}{L}} \sin(\omega_n z) \tag{7}$$

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where $\omega_n = \pi n/L$ with $n = 1, 2, 3, \ldots$. The total vacuum energy is therefore

$$E_0 = \frac{\pi}{2L} \sum_{n=1}^{\infty} n = -\frac{\pi}{24L} \tag{8}$$

when we use zeta-function regularization where $\zeta(-1) = -1/12$. Assuming a constant vacuum energy density, it is therefore $u = -\pi/24L^2$.

Let us now consider the partial contributions to the energy. The part coming from fluctuations in the electric field is given by the divergent sum

$$\mathcal{E}_E = \frac{1}{2L} \sum_{n=1}^{\infty} \omega_n \sin^2(\omega_n z) = \frac{1}{4L} \sum_{n=1}^{\infty} \omega_n (1 - \cos 2\omega_n z) \tag{9}$$

If we integrate over the 1-dimensional volume to obtain the total Casimir energy, the first term gives $-\pi/48L$ after zeta-function regularization. But in the last position-dependent part, we see that each term gives zero because of the imposed boundary conditions. We are thus left with a finite contribution to the total energy which is exactly one half of the full energy. The other half comes from the magnetic contribution.

Let us now instead derive a finite result for the energy density. It can be written as

$$\mathcal{E}_E = -\frac{\pi}{48L^2} - \frac{\pi}{8L^2} \frac{d}{d\theta} S(\theta) \tag{10}$$

where

$$S(\theta) = \sum_{n=1}^{\infty} \sin 2\theta n \tag{11}$$

is a divergent sum. Using again zeta-function regularization, it becomes $S_{reg}(\theta) = \frac{1}{2} \cot \theta$ and we have the regularized energy density

$$\mathcal{E}_E^{reg} = -\frac{\pi}{16L^2} \left(\frac{1}{3} - \frac{1}{\sin^2 \theta} \right) \tag{12}$$

This corresponds to the result in (3) giving the fluctuations in the electric field between plates. The last term gives the dependence on the position in the volume. It is positive definite and diverges when we approach the boundaries of the volume, i.e. the end points of the interval. As a consequence the corresponding integrated energy is also infinite. This is in sharp contrast to the result above where the integrated contribution from the position-dependent term gave zero.

The underlying problem is obviously the behaviour of the fluctuations near the boundaries where $\sin \theta = 0$. Then the function (11) is zero while S_{reg} is infinite. In order to better understand this apparent paradox, we can use a more physical regularization method based upon an exponential cutoff instead of the previous more mathematical approach using the

analytical continuation of the zeta-function. We thus define the sum (11) by the limit $S(\theta) = \lim_{\epsilon \to 0} S(\epsilon, \theta)$ of the convergent sum

$$S(\epsilon, \theta) = \sum_{n=1}^{\infty} e^{-\epsilon n} \sin 2\theta n = \frac{e^{-\epsilon} \sin 2\theta}{1 - 2e^{-\epsilon} \cos 2\theta + e^{-2\epsilon}} \tag{13}$$

Since it is the derivative of this sum which appears in the energy density (10), it is seen not to contribute to the integrated energy even with a non-zero cutoff. This is consistent with what we found using zeta-function regularization. However, the regularized energy density is found in the limit $\epsilon \to 0$ from

$$S(\epsilon, \theta) = \frac{1}{2} \cot \theta - \frac{1}{8} \frac{\cos \theta}{\sin^3 \theta} \epsilon^2 + \mathcal{O}(\epsilon^4) \tag{14}$$

As long as $\sin \theta > 0$, i.e. inside the boundaries, we can take the $\epsilon = 0$ and we recover the zeta-function result. But when we get sufficiently close to the boundaries, the expansion breaks down and we have in general a cutoff-dependent result. This should not come as a complete surprise since the energy density on the boundaries must be strongly dependent on the microscopic and physical properties of the material in the boundaries. But the welcome result is that the integrated energy is independent of the cutoff and the regularization method.

One can emulate the Euler-Heisenberg interaction in our one-dimensional system by considering the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{\alpha}{m^2} (\partial_\mu \phi)^4 \tag{15}$$

Here α is some small dimensionless constant and m a heavy mass. The Lagrangian is invariant under the field transformation $\phi \to \phi + const$ and thus describes massless particles. Treating the interaction in lowest order perturbation theory, one easily finds the resulting vacuum energy density after regularization to be

$$\mathcal{E} = -\frac{\pi}{24L^2} - \frac{\alpha \pi^2}{8m^2 L^4} \left(\frac{1}{18} + \frac{1}{\sin^4 \theta} \right) \tag{16}$$

Now it diverges near the endpoints where $\sin \theta \to 0$ and the integrated Casimir energy is infinite. However, if we instead first integrate the energy density and then regularize as above, we get the finite result

$$E_0 = -\frac{\pi}{24L} - \frac{\alpha \pi^2}{144 m^2 L^3} \tag{17}$$

The potentially divergent part now vanishes since $\zeta(-2) = 0$. We see that the total Casimir energy is furnished by just the constant part of energy density (16). Again there is no contribution from the position-dependent terms.

With this approach we can now calculate a finite, total Casimir energy for the photons between two plates and interacting via the Euler-Heisenberg interaction (2). Zeta-function

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regularization must now be combined with dimensional regularization for the transverse degrees of freedom. As a result we find again that the position-dependent part of the energy density (6) does not contribute to the integrated energy which now becomes[15]

$$E_0 = -\frac{\pi^2}{720L^3} - \frac{11\alpha^2\pi^4}{2^7 3^5 5^3 m^4 L^7} \tag{18}$$

Since the energy density between the plates can thus effectively be taken to be constant in calculating the total energy, it could also have been obtained more directly by requiring instead the field to be periodic between the plates with period $\beta = 2L$. This will then give a constant energy density $\mathcal{E} = E_0/L$ also for the interacting field. Letting $L \to 1/2T$ in the expression for $\mathcal{E}$ we thus have the free energy density for an interacting photon gas in thermal equilibrium at temperature T. The result is in agreement with a previous calculation by Barton using a semi-classical method[25].

References

[1] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[2] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).

[3] M.J. Sparnaay, Physica **24**, 751 (1958).

[4] J. Mehra, Physica **37**, 145 (1967).

[5] L.S. Brown and G.J. Maclay, Phys. Rev. **184**, 1272 (1969).

[6] J. Schwinger, L.L. DeRaad and K.A. Milton, Ann. Phys. (NY) **115**, 1 (1978).

[7] F. Ravndal and D. Tollefsen, Phys. Rev. **D40**, 4191 (1989).

[8] V.M. Mostepanenko and N.N. Trunov, Sov. Phys. Usp. **31**, 965 (1988).

[9] P.W. Milonni, *The Quantum Vacuum*, Academic Press, Boston, 1994.

[10] M. Bordag, D. Robaschik and E. Wieczorek, Ann. Phys. (NY) **165**, 192 (1985).

[11] She-Sheng Xue, Comm. Theor. Phys. **11**, 243 (1989).

[12] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1 and 2, Cambridge University Press, Cambridge, 1995.

[13] D. Kaplan, nucl-th/9506035, Lectures given at the 7th Summer School in Nuclear Physics Symmetries, 1995.

[14] C. Itzykson and J-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1980.

[15] X. Kong and F. Ravndal, in preparation.

[16] E. Euler, Ann. Phys. **26**, 398 (1936); E. Euler and W. Heisenberg, Z. Phys. **98**, 714 (1936).

[17] K. Scharnhorst, Phys. Lett. **B236**, 354 (1990); G. Barton, Phys. Lett. **B237**, 559 (1990); G. Barton and K. Scharnhorst, J. Phys. **A26**, 2037 (1993).

[18] C.A. Lütken and F. Ravndal, Phys. Rev. **A31**, 2082, (1985).

[19] D. Deutsch and P. Candelas, Phys. Rev. **D20**, 3063 (1979).

[20] K. Olaussen and F. Ravndal, Nucl. Phys. **B192**, 237 (1981); K. A. Milton, Phys. Lett. **B104**, 49 (1981).

[21] T.H. Boyer, Phys. Rev. **174**, 1764 (1968).

[22] C.M. Bender and P. Hays, Phys. Rev. **D14**, 2622 (1976).

[23] J. Schwinger, L.L. DeRaad and K.A. Milton, Ann. Phys. (NY) **115**, 388 (1978).

[24] P. Candelas, Ann. Phys. (NY) **143**, 241 (1982).

[25] G. Barton, Ann. Phys. (NY) **205**, 49 (1991).